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Existence of Three Positive Solutions for Some Second-Order Boundary Value Problems

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Abstract—In this paper, a new fixed-point theorem of functional type in a cone is established. With using the new fixed-point theorem and imposing growth conditions on the nonlinearity, the existence of three positive solutions for the boundary value problem

$$x''(t) + f\left(t, x(t), x'(t)\right) = 0, \qquad 0 < t < 1,$$
$$x(0) = x(1) = 0,$$

is obtained. Here $f : [0,1] \times [0,\infty) \times R \to [0,\infty)$ is continuous. Finally, an example is given to illustrate the importance of results obtained.

Keywords—Fixed-point theorem, Boundary value problem, Positive solution.

1. INTRODUCTION

In the past 20 years, there has been much attention focused on questions of positive solutions for diverse nonlinear ordinary differential equation, difference equation, and functional differential equation boundary value problems without dependence on the first-order derivative, see [1–8] and references therein. It is well known that the Guo-Krasnosel'skii fixed-point theorem in a cone [3,6] and Leggett-Williams fixed-point theorem [5] play an extremely important role in above study.

Recently, some new fixed-point theorems, for example, five functional fixed-point theorem [9,10] due to Avery, and fixed-point theorem of cone expansion and compression of functional type [11]

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due to Avery and Anderson, the twin fixed-point theorem [12] due to Avery and Henderson, a generalization of Leggett-Williams fixed-point theorem [13] due to Avery and Peterson, were applied to obtain some new existence and multiplicity results. All of them can be regarded as extensions of the Leggett-Williams fixed-point theorem and Guo-Krasnosel'skii fixed-point theorem.

For the other literature with respect to the existence of triple solutions, but not necessary positive solutions, we refer reader to [14–17]. The method used in [16] is the upper and lower solution method, and the method used in [14,15,17] is the innovative minimal method of the critical point theory which is due to Ricceri.

However, all the above works about positive solutions were done under the assumption that the first order derivative is not involved explicitly in the nonlinear term. On the other hand, to my best knowledge, there are very few work consider the multiplicity of positive solutions with dependence on derivatives. Motivated by all the above works, the aim of this paper is to generalize the Leggett-Williams fixed-point theorem so that some new multiplicity results could be obtained. Via the fixed-point theorem, we show the existence of triple positive solutions for the second-order two point boundary value problem

$$x''(t) + f(t, x(t), x'(t)) = 0, \qquad 0 < t < 1,$$
$$x(0) = x(1) = 0,$$

where $f : [0,1] \times [0,\infty) \times R \to [0,\infty)$ is continuous. Finally, we present an example to check our result.

2. A NEW FIXED-POINT THEOREM IN A CONE

In this section, we introduce a new fixed-point theorem in a cone which can be regarded as a generalization of the Leggett-Williams fixed-point theorem. The following result of fixed-point index theory is fundamental to our proofs.

LEMMA 2.1. *(See [3].) Let E be a Banach space and X be a retract of E, X_1 be a bounded convex retract of X, and $U \subset X$ be a nonempty open subset, such that $U \subset X_1$. If $T : X_1 \to X$ is completely continuous, $T(X_1) \subset X_1$, such that there is no fixed point of T in $X_1 \setminus U$, then $i(T, U, X) = 1$. In particular, if E is a real Banach space, $X \subset E$ is a nonempty bounded convex closed subset, and $T : X \to X$ is a completely continuous operator, then $i(T, X, X) = 1$.*

To state the new fixed-point theorem we require the following.

DEFINITION 2.1. *Let E be a Banach space over R. A nonempty convex closed set $P \subset E$ is said to be a cone provided that*

(i) *$au \in P$ for all $u \in P$ and all $a \geq 0$, and*
(ii) *$u, -u \in P$ implies $u = 0$.*

DEFINITION 2.2. *The map γ is said to be a nonnegative continuous concave functional on P provided that $\gamma : P \to [0,\infty)$ is continuous and*

$$\gamma(tx + (1-t)y) \geq t\gamma(x) + (1-t)\gamma(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$. Similarly, we say the map α is a nonnegative continuous convex functional on P provided that $\alpha : P \to [0,\infty)$ is continuous and

$$\alpha(tx + (1-t)y) \leq t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in P$ and $0 \leq t \leq 1$.

Suppose $\alpha, \beta : P \to [0,\infty)$ are two nonnegative continuous convex functionals satisfying

$$\|x\| \leq M \max\{\alpha(x), \beta(x)\}, \qquad \text{for } x \in P, \tag{2.1}$$

where M is a positive constant, and

$$\Omega = \{x \in P \mid \alpha(x) < r,\ \beta(x) < L\} \neq \emptyset, \qquad \text{for } r > 0, \quad L > 0. \tag{2.2}$$

With (2.1) and (2.2), Ω is a bounded nonempty open subset in P.

DEFINITION 2.3. *Let $r > a > 0$, $L > 0$ be given, $\alpha, \beta : P \to [0, \infty)$ be two nonnegative continuous convex functionals satisfying (2.1) and (2.2), and γ be a nonnegative continuous concave functional on the cone P. Define bounded convex sets*

$$\begin{aligned}
P(\alpha, r; \beta, L) &= \{x \in P \mid \alpha(x) < r, \beta(x) < L\},\\
\bar{P}(\alpha, r; \beta, L) &= \{x \in P \mid \alpha(x) \leq r, \beta(x) \leq L\},\\
P(\alpha, r; \beta, L; \gamma, a) &= \{x \in P \mid \alpha(x) < r, \beta(x) < L, \gamma(x) > a\},\\
\bar{P}(\alpha, r; \beta, L; \gamma, a) &= \{x \in P \mid \alpha(x) \leq r, \beta(x) \leq L, \gamma(x) \geq a\}.
\end{aligned}$$

We now give the fixed-point theorem.

THEOREM 2.1. *Let E be a Banach space, $P \subset E$ be a cone and $r_2 \geq d > b > r_1 > 0$, $L_2 \geq L_1 > 0$ be given. Assume that α, β are nonnegative continuous convex functionals on P, such that (2.1) and (2.2) are satisfied, γ is a nonnegative continuous concave functional on P, such that $\gamma(x) \leq \alpha(x)$ for all $x \in \bar{P}(\alpha, r_2; \beta, L_2)$ and let $T : \bar{P}(\alpha, r_2; \beta, L_2) \to \bar{P}(\alpha, r_2; \beta, L_2)$ be a completely continuous operator. Suppose*

(C1) $\{x \in \bar{P}(\alpha, d; \beta, L_2; \gamma, b) \mid \gamma(x) > b\} \neq \emptyset$, $\gamma(Tx) > b$, *for* $x \in \bar{P}(\alpha, d; \beta, L_2; \gamma, b)$,
(C2) $\alpha(Tx) < r_1, \beta(Tx) < L_1$, *for all* $x \in \bar{P}(\alpha, r_1; \beta, L_1)$,
(C3) $\gamma(Tx) > b$, *for all* $x \in \bar{P}(\alpha, r_2; \beta, L_2; \gamma, b)$ *with* $\alpha(Tx) > d$.

Then T has at least three fixed points x_1, x_2, and x_3 in $\bar{P}(\alpha, r_2; \beta, L_2)$. Further,

$$x_1 \in P(\alpha, r_1; \beta, L_1), \qquad x_2 \in \left\{\bar{P}(\alpha, r_2; \beta, L_2; \gamma, b) \mid \gamma(x) > b\right\}$$

and

$$x_3 \in \bar{P}(\alpha, r_2; \beta, L_2) \setminus \left(\bar{P}(\alpha, r_2; \beta, L_2; \gamma, b) \cup \bar{P}(\alpha, r_1; \beta, L_1)\right).$$

PROOF. Setting $U_1 = \{x \mid x \in P(\alpha, r_1; \beta, L_1)\}$, $U_2 = \{x \mid x \in \bar{P}(\alpha, r_2; \beta, L_2; \gamma, b) \mid \gamma(x) > b\}$. With assumptions on α, β and γ, U_1 and U_2 are disjoint bounded nonempty open subsets of $\bar{P}(\alpha, r_2; \beta, L_2)$. By (C2), there is $T(\bar{U}_1) \subset U_1$, then Lemma 2.1 implies

$$i\left(T, U_1, \bar{P}(\alpha, r_2; \beta, L_2)\right) = 1. \tag{2.3}$$

Similarly, as $T : \bar{P}(\alpha, r_2; \beta, L_2) \to \bar{P}(\alpha, r_2; \beta, L_2)$ is completely continuous, with Lemma 2.1 there holds

$$i\left(T, \bar{P}(\alpha, r_2; \beta, L_2), \bar{P}(\alpha, r_2; \beta, L_2)\right) = 1. \tag{2.4}$$

Now, let's prove $Tx \neq x$, for all $x \in \partial U_2$. In fact, if $x_0 \in \partial U_2$, such that $Tx_0 = x_0$, then there is either

(i) $\gamma(x_0) = b$ and $\alpha(x_0) > d$, or
(ii) $\gamma(x_0) = b$ and $x_0 \in \bar{P}(\alpha, d; \beta, L_2; \gamma, b)$.

In Case (i), there is $\alpha(Tx_0) = \alpha(x_0) > d$, which combine (C3) yields $\gamma(x_0) = \gamma(Tx_0) > b$, it is a contradiction.

In Case (ii), by (C1), $\gamma(x_0) = \gamma(Tx_0) > b$, a contradiction, too.

Thus, $Tx \neq x$, for all $x \in \partial U_2$, and $i(T, U_2, \bar{P}(\alpha, r_2; \beta, L_2))$ is well defined.

Choose $z_0 \in \bar{P}(\alpha, d; \beta, L_2; \gamma, b)$ satisfying $\gamma(z_0) > b$. Let

$$h(t, x) = tz_0 + (1 - t)Tx.$$

Clearly, $h : [0,1] \times \bar{U}_2 \to \bar{P}(\alpha, r_2; \beta, L_2)$ is completely continuous. Suppose there exist $(t_0, x_0) \in [0,1] \times \partial U_2$, such that $h(t_0, x_0) = x_0$, then $\gamma(x_0) = b$. We divide the situation to two cases: $\alpha(Tx_0) > d$ and $\alpha(Tx_0) \le d$. If $\alpha(Tx_0) > d$, by Condition (C3), there is $\gamma(Tx_0) > b$, and

$$\begin{aligned}\gamma(x_0) &= \gamma(h(t_0, x_0)) \\ &= \gamma(t_0 z_0 + (1-t_0)Tx_0) \\ &\ge t_0\gamma(z_0) + (1-t_0)\gamma(Tx_0) > b,\end{aligned}$$

it is a contradiction; if $\alpha(Tx_0) \le d$,

$$\begin{aligned}\alpha(x_0) &= \alpha(t_0 z_0 + (1-t_0)Tx_0) \\ &\le t_0\alpha(z_0) + (1-t_0)\alpha(Tx_0) \le d,\end{aligned}$$

therefore, $x_0 \in \bar{P}(\alpha, d; \beta, L_2; \gamma, b)$, by Condition (C1), there holds $\gamma(Tx_0) > b$, and

$$\begin{aligned}\gamma(x_0) &= \gamma(t_0 z_0 + (1-t_0)Tx_0) \\ &\ge t_0\gamma(z_0) + (1-t_0)\gamma(Tx_0) > b,\end{aligned}$$

a contradiction, too. Thus, we have

$$h(t,x) \neq x, \qquad \text{for } (t,x) \in [0,1] \times \partial U_2.$$

According to the homotopy invariance of fixed-point index,

$$i\left(T, U_2, \bar{P}(\alpha, r_2; \beta, L_2)\right) = i\left(z_0, U_2, \bar{P}(\alpha, r_2; \beta, L_2)\right) = 1. \tag{2.5}$$

With the additivity of fixed-point index and (2.3)–(2.5), there is

$$\begin{aligned}&i\left(T, \bar{P}(\alpha, r_2; \beta, L_2) \setminus \left(\overline{U_1 \cup U_2}\right), \bar{P}(\alpha, r_2; \beta, L_2)\right) \\ &\quad = i\left(T, \bar{P}(\alpha, r_2; \beta, L_2), \bar{P}(\alpha, r_2; \beta, L_2)\right) \\ &\qquad - i\left(T, U_1, \bar{P}(\alpha, r_2; \beta, L_2)\right) - i\left(T, U_2, \bar{P}(\alpha, r_2; \beta, L_2)\right) \\ &\quad = 1 - 1 - 1 = -1.\end{aligned} \tag{2.6}$$

Therefore, there exist $x_1 \in U_1$, $x_2 \in U_2$, $x_3 \in \bar{P}(\alpha, r_2; \beta, L_2) \setminus (\overline{U_1 \cup U_2})$, such that

$$Tx_i = x_i, \qquad i = 1, 2, 3.$$

The proof is complete. ∎

REMARK 2.1. Theorem 2.1 can be regarded as a generalization of Leggett-Williams fixed-point theorem of cone. In fact, choose $\alpha(x) = \|x\|$, $\beta(x) \equiv 0$, we can get Leggett-Williams fixed-point theorem of cone. The proof of Theorem 2.1 is similar to that of Leggett-Williams fixed-point theorem. Here, inspired by the works of Avery *et al.*, we first introduce two convex functionals, α and β, take instead of norm in defining convex subsets, the technique has been also used by the authors to generalize the Krasnosel'skii fixed-point theorem. The generalization seems to be useful in determine the existence of triple solutions of nonlinear differential equation boundary value problems with dependence on the first-order derivative.

3. MULTIPLICITY RESULTS OF POSITIVE SOLUTIONS

In this section, we give some application of Theorem 2.1 to dwell upon importance of the new fixed-point theorem in a cone.

We are concerned with the existence of triple positive solutions for the second-order two point boundary value problem

$$x''(t) + f\left(t, x(t), x'(t)\right) = 0, \qquad 0 < t < 1, \tag{3.1}$$

$$x(0) = x(1) = 0, \tag{3.2}$$

where $f : [0,1] \times [0,\infty) \times R \to [0,\infty)$ is continuous.

Let $X = C^1[0,1]$ be endowed with the ordering $x \le y$ if $x(t) \le y(t)$, for all $t \in [0,1]$, and the maximum norm, $\|x\| = \max\{\max_{0\le t\le 1} |x(t)|, \max_{0\le t\le 1} |x'(t)|\}$. From the fact $x''(t) = -f(t,x,x') \le 0$, we know that x is concave on $[0,1]$. So, define the cone $P \subset X$ by

$$P = \{x \in X \mid x(t) \ge 0, x \text{ is concave on } [0,1]\} \subset X.$$

Define functionals

$$\alpha(x) = \max_{0\le t\le 1} |x(t)|, \quad \beta(x) = \max_{0\le t\le 1} |x'(t)|, \quad \gamma(x) = \min_{1/4\le t\le 3/4} |x(t)|, \qquad \text{for } x \in P.$$

Then $\alpha, \beta, \gamma : P \to [0,\infty)$ are three continuous nonnegative functionals such that $\|x\| = \max\{\alpha(x), \beta(x)\}$, and (2.1), (2.2) hold; α, β are convex, γ is concave and there holds $\gamma(x) \le \alpha(x)$, for all $x \in P$. Denote $G(t,s)$ the Green's function for boundary value problem

$$-x''(t) = 0, \qquad 0 < t < 1,$$
$$x(0) = x(1) = 0.$$

then $G(t,s) \ge 0$ for $0 \le t, s \le 1$ and

$$G(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1. \end{cases}$$

Let

$$\delta = \min\left\{ \int_{1/4}^{3/4} G\left(\frac{1}{4}, s\right) ds, \int_{1/4}^{3/4} G\left(\frac{3}{4}, s\right) ds \right\} = \frac{1}{16}.$$

THEOREM 3.1. *Suppose there exist constants $r_2 \ge 4b > b > r_1 > 0, L_2 \ge L_1 > 0$ such that $b/\delta \le \min\{8r_2, 2L_2\}$ and the following assumptions hold:*

(A1) *$f(t,u,v) < \min\{8r_1, 2L_1\}$, for $(t,u,v) \in [0,1] \times [0,r_1] \times [-L_1, L_1]$;*

(A2) *$f(t,u,v) > b/\delta$, for $(t,u,v) \in [1/4,3/4] \times [b,4b] \times [-L_2, L_2]$;*

(A3) *$f(t,u,v) \le \min\{8r_2, 2L_2\}$, for $(t,u,v) \in [0,1] \times [0,r_2] \times [-L_2, L_2]$.*

Then, the boundary value problem (3.1),(3.2) has at least three positive solutions x_1, x_2, and x_3 satisfying

$$\max_{0\le t\le 1} x_1(t) \le r_1, \qquad \max_{0\le t\le 1} |x_1'(t)| \le L_1;$$
$$b < \min_{1/4\le t\le 3/4} x_2(t) \le \max_{0\le t\le 1} x_2(t) \le r_2, \max_{0\le t\le 1} |x_2'(t)| \le L_2;$$
$$\max_{0\le t\le 1} x_3(t) \le 4b, \qquad \max_{0\le t\le 1} |x_3'(t)| \le L_2.$$

PROOF. Problem (3.1),(3.2) has a solution $x = x(t)$ if and only if x solves the operator equation

$$x(t) = Tx(t) := \int_0^1 G(t,s) f\left(s, x(s), x'(s)\right) ds.$$

It is well known that $T : P \to P$ is completely continuous.

We now show that all the conditions of Theorem 2.1 are satisfied.

If $x \in \bar{P}(\alpha, r_2; \beta, L_2)$, then there is $\alpha(x) \le r_2$, $\beta(x) \le L_2$ and Assumption (A3) implies $f(t, x(t), x'(t)) \le \min\{8r_2, 2L_2\}$. Consequently,

$$\begin{aligned}\alpha(Tx) &= \max_{t\in[0,1]} \left| \int_0^1 G(t,s) f(s, x(s), x'(s))\, ds \right| \\ &\le 8r_2 \cdot \max_{t\in[0,1]} \int_0^1 G(t,s)\, ds = r_2.\end{aligned}$$

For $x \in P$, there is $Tx \in P$, then Tx is concave on $[0,1]$, and $\max_{t\in[0,1]} |(Tx)'(t)| = \max\{|(Tx)'(0)|, |(Tx)'(1)|\}$, so

$$\begin{aligned}\beta(Tx) &= \max_{t\in[0,1]} |(Tx)'(t)| \\ &= \max_{t\in[0,1]} \left| -\int_0^t sf(s, x(s), x'(s))\, ds + \int_t^1 (1-s) f(s, x(s), x'(s))\, ds \right| \\ &= \max\left\{ \int_0^1 (1-s) f(s, x(s), x'(s))\, ds, \int_0^1 sf(s, x(s), x'(s))\, ds \right\} \\ &\le 2L_2 \cdot \max\left\{ \int_0^1 (1-s)\, ds, \int_0^1 s\, ds \right\} \\ &\le 2L_2 \cdot \frac{1}{2} = L_2.\end{aligned}$$

Hence, $T : \bar{P}(\alpha, r_2; \beta, L_2) \to \bar{P}(\alpha, r_2; \beta, L_2)$. In the same way, if $x \in \bar{P}(\alpha, r_1; \beta, L_1)$, then Assumption (A1) yields $f(t, x(t), x'(t)) < \min\{8r_1, 2L_1\}, 0 \le t \le 1$. As in the argument above, we can obtain that $T : \bar{P}(\alpha, r_1; \beta, L_1) \to P(\alpha, r_1; \beta, L_1)$. Therefore, Condition (C2) of Theorem 2.1 is satisfied.

To check Condition (C1) of Theorem 2.1, we choose $x(t) = 4b$, $0 \le t \le 1$. It is easy to see that $x(t) = 4b \in \bar{P}(\alpha, 4b; \beta, L_2; \gamma, b)$ and $\gamma(x) = \gamma(4b) > b$, and so $\{x \in \bar{P}(\alpha, 4b; \beta, L_2; \gamma, b) \mid \gamma(x) > b\} \ne \emptyset$. Hence, if $x \in \bar{P}(\alpha, 4b; \beta, L_2; \gamma, b)$, then $b \le x(t) \le 4b$ for $1/4 \le t \le 3/4$. From Assumption (A2), we have $f(t, x(t), x'(t)) \ge b/\delta$ for $1/4 \le t \le 3/4$, and by the conditions of γ and the cone P, we have to distinguish two cases,

(i) $\gamma(Tx) = (Tx)(1/4)$, and
(ii) $\gamma(Tx) = (Tx)(3/4)$.

In Case (i), we have

$$\begin{aligned}\gamma(Tx) &= (Tx)\left(\frac{1}{4}\right) \\ &= \int_0^1 G\left(\frac{1}{4}, s\right) f(s, x(s), x'(s))\, ds \\ &> \frac{b}{\delta} \cdot \int_{1/4}^{3/4} G\left(\frac{1}{4}, s\right) ds \\ &\ge b.\end{aligned}$$

In Case (ii), we have

$$\begin{aligned}\gamma(Tx) &= (Tx)\left(\frac{3}{4}\right) \\ &= \int_0^1 G\left(\frac{3}{4}, s\right) f(s, x(s), x'(s))\, ds \\ &> \frac{b}{\delta} \cdot \int_{1/4}^{3/4} G\left(\frac{3}{4}, s\right) ds \\ &\ge b;\end{aligned}$$

i.e.,

$$\gamma(Tx) > b, \qquad \text{for all } x \in \bar{P}(\alpha, 4b; \beta, L_2; \gamma, b).$$

This show that Condition (C1) of Theorem 2.1 is satisfied. We finally show that (C3) of Theorem 2.1 also holds. Suppose that $x \in \bar{P}(\alpha, r_2; \beta, L_2; \gamma, b)$ with $\alpha(Tx) > 4b$. Then, by the conditions of γ and $Tx \in P$, we have

$$\begin{aligned}\gamma(Tx) &= \min_{1/4 \le t \le 3/4} (Tx)(t) \\ &\ge \frac{1}{4} \cdot \max_{0 \le t \le 1} (Tx)(t) \\ &= \frac{1}{4} \cdot \alpha(Tx) \\ &> \frac{1}{4} \cdot 4b = b.\end{aligned}$$

So, Condition (C3) of Theorem 2.1 is satisfied. Therefore, an application of Theorem 2.1 imply there exist three solutions of problem (3.1),(3.2), such that

$$x_1 \in P(\alpha, r_1; \beta, L_1), \qquad x_2 \in \left\{\bar{P}(\alpha, r_2; \beta, L_2; \gamma, b) \mid \gamma(x) > b\right\}$$

and

$$x_3 \in \bar{P}(\alpha, r_2; \beta, L_2) \setminus \left(\bar{P}(\alpha, r_2; \beta, L_2; \gamma, b) \cup \bar{P}(\alpha, r_1; \beta, L_1)\right).$$

In addition, as x_3 satisfies $\alpha(x_3) \le 4\gamma(x_3)$, there is $\max_{0 \le t \le 1} x_3(t) \le 4b$. The proof is complete. ∎

REMARK 3.1. With Theorem 2.1, we have the result $\max_{0 \le t \le 1} x_3(t) \le r_2, \min_{1/4 \le t \le 3/4} x_3(t) < b$. However, for problem (3.1),(3.2), the functional α and γ holds additional relation

$$\gamma(x) = \min_{1/4 \le t \le 3/4} x(t) \ge \frac{1}{4} \max_{0 \le t \le 1} x(t) = \frac{1}{4} \alpha(x), \qquad \text{for } x \in P,$$

so, we can acquire the better result $\max_{0 \le t \le 1} x_3(t) \le 4b$.

From Theorem 3.1, we see that, when assumptions like (A1)–(A3) are appropriately imposed on f, we can obtain any number of positive solutions of problem (3.1),(3.2). To be more precise, we have the following conclusion.

COROLLARY 3.1. *Suppose that there exist constants* $0 < r_1 < b_1 < 4b_1 \le r_2 < b_2 < 4b_2 \le \dots \le r_n$, $0 < L_1 \le L_2 \le \cdots \le L_{n-1}, n \in N$, *such that* $b_i/\delta \le \min\{8r_{i+1}, 2L_{i+1}\}$ *for* $1 \le i \le n-1$ *and the following conditions are satisfied:*

(E1) $f(t, u, v) < \min\{8r_i, 2L_i\}$, *for* $(t, u, v) \in [0, 1] \times [0, r_i] \times [-L_i, L_i], 1 \le i \le n$;

(E2) $f(t, u, v) > b_i/\delta$, *for* $(t, u, v) \in [1/4, 3/4] \times [b_i, 4b_i] \times [-L_{i+1}, L_{i+1}], 1 \le i \le n-1$.

Then problem (3.1)–(3.2) admits at least $2n - 1$ *positive solutions.*

PROOF. When $n = 1$, it follows from Condition (E1) that $T : \bar{P}(\alpha, r_1; \beta, L_1) \to P(\alpha, r_1; \beta, L_1) \subset \bar{P}(\alpha, r_1; \beta, L_1)$, which means that at least one fixed point $x_1 \in P(\alpha, r_1; \beta, L_1)$ by the Schauder fixed-point theorem. When $n = 2$, it is clear that Theorem 3.1 holds (with $c_1 = r_2$). Then we can obtain at least three positive solutions x_2, x_3, and x_4. Following this way, we finish the proof by the induction method. ∎

Finally, we present an example to check our results.

Consider the boundary value problem

$$x''(t) + f(t, x(t), x'(t)) = 0, \qquad 0 < t < 1, \tag{3.3}$$

$$x(0) = x(1) = 0, \tag{3.4}$$

where

$$f(t,u,v)=\begin{cases}\sin t+\dfrac{9}{2}u^3+\left(\dfrac{|v|}{300}\right)^3, & \text{for } u\le 8,\\ \sin t+\dfrac{9}{2}(9-u)u^3+\left(\dfrac{|v|}{300}\right)^3, & \text{for } 8\le u\le 9,\\ \sin t+\dfrac{9}{2}(u-9)u^3+\left(\dfrac{|v|}{300}\right)^3, & \text{for } 9\le u\le 10,\\ \sin t+4500+\left(\dfrac{|v|}{300}\right)^3, & \text{for } u\ge 10.\end{cases}$$

Choose $r_1=1$, $b=2$, $r_2=1000$, $L_1=10$, $L_2=3000$, we note $\min\{8r_1,2L_1\}=8$, $b/\delta=32$, $\min\{8r_2,2L_2\}=6000$. Consequently, $f(t,u,v)$ satisfy

$$\begin{aligned}&f(t,u,v)<8, && \text{for } 0\le t\le 1, && 0\le u\le 1, && -10\le v\le 10;\\ &f(t,u,v)>32, && \text{for } \frac{1}{4}\le t\le\frac{3}{4}, && 2\le u\le 8, && -3000\le v\le 3000;\\ &f(t,u,v)<6000, && \text{for } 0\le t\le 1, && 0\le u\le 1000, && -3000\le v\le 3000.\end{aligned}$$

Then all assumptions of Theorem 3.1 hold. Thus, with Theorem 3.1, problem (3.3),(3.4) has at least three positive solutions x_1, x_2, x_3, such that

$$\max_{0\le t\le 1} x_1(t)\le 1, \qquad \max_{0\le t\le 1}|x_1'(t)|\le 10;$$

$$2<\min_{1/4\le t\le 3/4} x_2(t)\le \max_{0\le t\le 1} x_2(t)\le 1000,\ \max_{0\le t\le 1}|x_2'(t)|\le 3000;$$

$$\max_{0\le t\le 1} x_3(t)\le 8, \qquad \max_{0\le t\le 1}|x_3'(t)|\le 3000.$$

REMARK 3.2. The early results, see [1,5,9,11–13,17], for example, are not applicable to the above problem. In conclusion, we see that the nonlinear term is involved in first derivative explicitly. Meanwhile, as the nonlinear term isn't satisfy the Nagumo-style condition, so the classic upper and lower solution method, see [16], cannot be used to solve this example.

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